Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Markov Chains (M2)

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Max. points : 40. **Time Limit : 3 hours.**

Answer any four questions. All questions carry equal points.

Give complete proofs. Please cite/quote appropriate results from class or assignments properly.

1. *Lazy random walk on the discrete torus :* Consider the lazy random walk on the d-dimensional discrete torus $\mathbb{Z}_n^d = \{0, \ldots, n-1\}^d$. $\mathbb{Z}_n^d$ can be described as the following graph : $x \sim y$ if there exists $j \in \{1, \ldots, d\}$ such that $x_i = y_i$ for all $i \neq j$ and $x_j \in \{y_j +_n 1, y_j -_n 1\}$ where $+_n, -_n$ denote addition and subtraction modulo n. Show that $t_{mix} \leq d^2 n^2$ for the lazy random walk on $\mathbb{Z}_n^d$.

2. *Lazy random walk on the hypercube :* Consider the lazy random walk on the n-dimensional hypercube $H_n = \{-1, 1\}^n$. Compute the eigenfunctions, eigenvalues, spectral gap and absolute spectral gap of this Markov chain. Using the above computations, derive upper and lower bounds on the mixing time t_{mix} of the Markov chain.

3. *Winning Streak/ Success Runs :* Consider the following process X_n on $\mathbb{N} \cup \{0\}$: Suppose the process $X_n = i \in \mathbb{N} \cup \{0\}$, a coin C_i (whose probability of heads is p_i) is tossed and if it is heads, the process goes up by one step. If the coin lands tails, the process returns back to origin. Note that there are countably many (possibly) distinct coins C_i each of which is associated with the corresponding state i. Denote by $\bar{p} = \{p_i\}_{i \geq 0}$, the sequence of 'head' probabilities of the coins C_i's.

   Specify the transition matrix of the chain and give (sufficient and necessary) conditions on $\bar{p}$ for the irreducibility of X_n. Analyze the recurrence, positive recurrence and transience properties of the above chain i.e., characterize the values of $\bar{p}$ for which the properties hold.

4. *Bounded Winnning Streak :* Let X_n be the Markov chain defined as above. Define a new Markov chain $Y_n = \min\{X_n, K\}$ for some strictly positive integer K. Compute the transition matrix and stationary distribution of the chain. Compute the transition matrix $\hat{P}$ of the reversed chain and show that its stationary distribution is same as that of the original chain. Show that $\hat{P}^n_\mu(.)$ is the stationary distribution for any $n > K$ and any initial distribution μ.

5. *Discrete Queuing System :* Consider the following queueing model. At time instant n, Y_n customers are served and leave the queue. If there are less than Y_n customers in the queue, all of them are served. Further, Z_n new customers join the queue. Let X_n be the number of customers at the end of time instant n. Assume that $Z_n, n \geq 1$ and $Y_n, n \geq 1$ are both independent i.i.d sequences distributed as Z and Y respectively. Here Z, Y are $\mathbb{N}$-valued random variables such that $E(Y^2), E(Z^2) < \infty$ and $E(Y) - E(Z) \neq 0$. Assuming irreducibility, analyse the recurrence and transience properties of X_n in terms of the distributions of Z, Y.